Quantum Friction

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This is a slightly extended version of the "long" short talk given at the Lisbon school "Topology of strongly correlated systems" in the Fall of 2000. After a pedagogical introduction into the field of dissipative quantum mechanics and its methods I explain recent results concerning the problem of localization on attractive potentials in the presence of friction.

1 Introduction

In recent years there has been a continued interest in what is known as dissipative quantum mechanics.[1] To eliminate a possible confusion one should bear in mind that this is not a new type of quantum theory, but rather the good old quantum mechanics applied to the set of mathematically well-defined problems. Speaking generally, one can say that the dissipative quantum mechanics is concerned with problems where the system of interest is coupled to its environment. The goal is then to formulate a closed description for the system, "integrating out" the environmental degrees of freedom. The word "dissipative" comes from the fact that in most of the situations the coupling to the environment leads to energy exchange. But, in fact, the latter is not a necessary consequence of the former. In a wider context, one calls this set of problems the theory of open quantum systems or the theory of decoherence.[2]

In this talk, my aim will be to give an elementary introduction to one special (and relatively simple) case of the theory where the system of interest is a single particle and the environment is such that in the classical limit (say, at high temperature) its influence can be described by a friction force that is linear with the particle's velocity:

$$f = -\gamma \frac{dx}{dt}. \tag{1}$$

Besides dissipative single-particle problems, this case is also applicable to some dissipative many-particle systems where x plays the role of a collective coordinate. I will use the Lamb model,[3] which reproduces Eq. (1), to derive the effective Caldeira-Leggett action.[4] After applying this approach to two sample problems I will end with a result that I recently obtained concerning the problem of localization on attractive potentials.[5] There is vast literature on the subject of dissipative quantum mechanics and it is simply impossible to give all relevant references; instead I refer the reader to a very good book by Weiss.[1]

2 Friction and strings

Our first goal is to find a model that reproduces Eq. (1) in the classical limit. Consider an infinite classical elastic string. Let $\phi(s,t)$ be its displacement. Consider also a force $f(t)$ that is pulling the string at $s=0$. The equations of motion are:

$$\sigma\left(\partial_t^2 - c^2\partial_s^2\right)\phi(s,t) = f(t)\delta(s), \tag{2}$$

where σ is the mass per unit length of the string and c is the speed of sound in the string. The solution of Eq. (2) is:

$$\phi(s,t) = \frac{1}{2\sigma c}\int\limits_{-\infty}^{t-|s|/c} f(t')dt'. \tag{3}$$

An unperturbed state at $t=-\infty$ is assumed. In particular:

$$\phi(0,t) = \frac{1}{2\sigma c}\int\limits_{-\infty}^{t} f(t')dt', \tag{4}$$

$$f(t) = 2\sigma c\,\partial_t\phi(0,t). \tag{5}$$

If one then associates $\phi(0,t)=x$ and defines $\gamma = 2\sigma c$, Eq. (1) is reproduced. Thus if one concentrates on the dynamics of a particle which is placed on a string one finds an effective friction force of the form Eq. (1) which comes from the rest of the string. If one then assigns inertial mass m to the particle and puts it in a potential energy landscape $V(x)$, one obtains the Lamb model.[3]

3 Quantizing friction

The model of the previous section is well-understood on the classical level: indeed, the dynamics of the particle is completely determined by the equations of motion with the friction force. The solution $x(t)$ of these equations is everything one needs to know about the system. In the quantum case, however, the situation is more complicated. As is well-known, the complete information about the dynamics of a quantum system is contained in the density matrix. The time-dependence of the particle-plus-string density matrix is given by a unitary evolution operator. For the particle alone, however, no unitary evolution operator exists since the particle's energy is not conserved (i. e. no Hamiltonian system can be constructed). Thus one has to start with the full particle-plus-string system and then try to integrate out the string variables.

Consider the partition function of the entire system at temperature T=0. Its imaginary time path-integral representation is:

$$Z = \int \mathcal{D}x\mathcal{D}\phi\, e^{-S} \delta(\phi(0,\tau) - x(\tau)), \tag{6}$$

$$S = \int d\tau \left[\frac{m}{2}(\partial_\tau x)^2 + V(x)\right] + \frac{\sigma}{2}\int ds\, d\tau\, \left[(\partial_\tau \phi)^2 + c^2(\partial_s\phi)^2\right]. \tag{7}$$

In Eq. (6) the δ-function constraint ensures that the particle sits on the string. Using an integral representation of the δ-function and observing that the string action is quadratic one can integrate out the string variable ϕ to obtain an effective action description:

$$Z = \int \mathcal{D}x\, e^{-S_{\text{eff}}}, \tag{8}$$

$$S_{\text{eff}} = \int d\tau \left[\frac{m}{2}(\partial_\tau x)^2 + V(x)\right] + \frac{\gamma}{4\pi}\int\int d\tau\, d\tau' \frac{[x(\tau) - x(\tau')]^2}{[\tau - \tau']^2}, \tag{9}$$

where the friction constant $\gamma = 2\sigma c$, as before. This result was first obtained by Caldeira and Leggett.[4] Thus, friction is described by a time-non-local quadratic action. In general, non-locality is difficult to treat, but in some cases it is easy to understand its effect qualitatively:

Example 1: Dissipative quantum tunneling.[4] Consider a short-range attractive potential $V_0(x)$, centered at $x = 0$ with a bound state of energy $-\epsilon_0$ and biased by an external potential $-fx$. For $\gamma = 0$, the tunneling exponent is given by one-instanton approximation:

$$\Gamma_0 \sim e^{-S_0}, \tag{10}$$

$$S_0 = \int d\tau \left[\frac{m}{2}(\partial_\tau x_0)^2 + V(x_0)\right], \tag{11}$$

where x_0 is the classical instanton solution that connects points $x_i = 0$ and $x_f = \epsilon_0/f$. For small γ one can estimate the effect of friction on the tunneling probability by treating the γ-term in the action perturbatively. This gives:

$$\Gamma \approx \Gamma_0 \exp\left\{-\frac{\gamma}{4\pi}\int\int d\tau\, d\tau' \frac{[x_0(\tau) - x_0(\tau')]^2}{[\tau - \tau']^2}\right\}. \tag{12}$$

Observing that the integral in the exponent of Eq. (12) is invariant under time-rescaling (while $x_0(\tau)$ is not) one obtains the following estimate:

$$\Gamma \approx \Gamma_0 e^{-b\gamma x_f^2}, \qquad b \sim 1. \tag{13}$$

This is precisely the result obtained by Caldeira and Leggett for the suppression of tunneling by dissipation.

Example 2: Localization in a periodic potential.[6] Consider now a periodic potential $V(x)$ with a period a. The sum over histories in Eq. (8) will involve histories in which $x(\tau)$ stays in the same minimum of $V(x)$ as well as those in which $x(\tau)$ jumps from one minimum to another. Jumps from left to right are called instantons and those from right to left are called anti-instantons. Integrating the γ-term in Eq. (9) twice by parts one obtains:

$$S_\gamma = \frac{\gamma}{2\pi} \int \int d\tau \, d\tau' \, \frac{dx}{d\tau} \frac{dx}{d\tau'} \ln \frac{|\tau - \tau'|}{\tau_0}, \tag{14}$$

$$\frac{dx}{d\tau} = \sum_{j=1}^{2N} e_j \delta(\tau - \tau_j). \tag{15}$$

Here τ_0 is the instanton width and the e_j's are instanton "charges": $+a$ for an instanton and $-a$ for an anti-instanton. In the partition function, only the states with equal numbers of instantons and anti-instantons enter (trace is taken i. e. particle's paths are closed). Thus, the problem is equivalent to finding the partition function of the one-dimensional neutral plasma of logarithmically interacting charges. There is a phase transition in this problem as the friction constant γ is changed; it occurs at the critical value given by:

$$\frac{\gamma_c a^2}{2\pi} = 1. \tag{16}$$

Physically, for $\gamma > \gamma_c$ the attraction between instantons and anti-instantons becomes so strong that the instanton-anti-instanton pairs collapse. The particle no longer tunnels between the minima of the potential and is localized in one of them.

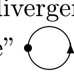
4 Localization on short-range potentials

Motivated by the problem of vortex pinning in the presence of friction, I have considered the problem of localization on attractive short-range potentials in dissipative quantum mechanics. Consider a particle of mass m that moves in d dimensions in the presence of an attractive short-range potential $V(\mathbf{x}) = -\alpha\delta(\mathbf{x})$. In the classical (high temperature) limit the particle experiences the action of a friction force that is linear in the particle's velocity: $\mathbf{f} = -\gamma\dot{\mathbf{x}}$. Then the $T = 0$ effective action description is given by Eqs. (8,9). Without the potential, the action is quadratic and the problem is solvable exactly. It describes the so-called quantum Brownian motion: the particle diffuses away from its original position. The question then can be formulated as follows: is the potential a relevant (in the renormalization group sense) perturbation? In other words, is the potential going to localize the particle or it will still be diffusing away to infinity? Below I will give a qualitative answer which, while lacking some rigor, best clarifies the situation; for a rigorous treatment of the problem see.[5]

To understand the more complicated dissipative case let's first try to understand what happens in the non-dissipative case where $\gamma = 0$. Let's start by writing down the imaginary time Green's function of the free particle Schrödinger equation:

$$G(\mathbf{x}_i, 0; \mathbf{x}_f, \tau) = \frac{1}{(2\pi R_\tau)^{\frac{d}{2}}} \exp\left\{-\frac{(\mathbf{x}_f - \mathbf{x}_i)^2}{2R_\tau}\right\}, \tag{17}$$

$$R_\tau = \frac{1}{d}\langle(\mathbf{x}(\tau) - \mathbf{x}(0))^2\rangle = 2\int \frac{d\omega}{2\pi}\frac{1-\cos\omega\tau}{m\omega^2} = \frac{\tau}{m}, \tag{18}$$

where m is the mass of the particle and R_τ describes how quickly the width of a wave packet grows with τ. Then, treating an attractive short-range potential $V(\mathbf{x}) = -\alpha\delta(\mathbf{x})$ as a perturbation, we see that its relevance in the renormalization group sense is determined by the infra-red divergence of the following integral (which corresponds to the so-called "bubble"  diagram):

$$\int^{\Lambda} G(0,0;0,\tau)d\tau \sim \int^{\Lambda} \frac{d\tau}{R_\tau^{\frac{d}{2}}} \sim \begin{cases} \Lambda^{\frac{1}{2}}, & d = 1 \rightarrow \quad \text{localization;} \\ \ln\Lambda, & d = 2 \rightarrow \ \substack{\text{exponentially}-\text{weak} \\ \text{localization;}} \\ \text{finite}, & d \geq 3 \rightarrow \ \text{no localization.} \end{cases} \tag{19}$$

So, the perturbation is relevant in $d = 1$, marginally relevant in $d = 2$ and irrelevant in $d \geq 3$. This reproduces the well-known quantum-mechanical results[7] on localized states in shallow potentials. Thus, whether the particles is localized by an attractive potential depends on how quickly $R_\tau^{\frac{d}{2}}$ grows with τ. In high dimensions it grows quickly enough so that the particle manages to escape, but in dimensions $d = 1, 2$ the growth is slower and the potential attraction overcomes the spreading of the wave packet.

Returning to our dissipative case, we need to evaluate $R_\tau = d^{-1}\langle(\mathbf{x}(\tau) - \mathbf{x}(0))^2\rangle$. Its behavior in the relevant long-time limit can be found directly from Eq. (9) (with $V(x) \equiv 0$) and is given by:

$$R_{\tau\to\infty} = 2\int \frac{d\omega}{2\pi}\frac{1-\cos\omega\tau}{\gamma|\omega|} = \frac{2}{\pi\gamma}\ln\frac{\gamma\tau}{m}. \tag{20}$$

We see that R_τ of a particle in the dissipative environment grows so slowly that the integral in Eq. (19) infra-red diverges for any d. It means that, contrary to the pure quantum case, an arbitrarily weak attractive short-range potential localizes such particle particle in any number of dimensions. A simple scaling analysis of Eq. (9) yields the energy of the localized state and its localization length when the strength of the potential α is small:

$$E_0 \sim -\alpha\gamma^{\frac{d}{2}}, \quad (21)$$

$$l \sim \gamma^{-\frac{1}{2}}. \quad (22)$$

It turns out that the correct results [5] differ from Eqs. (21,22) only by logarithmic factors. Comparing these formulas with their counterparts in pure quantum mechanics,[7] one sees that even in dimensions 1 and 2, where localization occurs both in the dissipative and non-dissipative problems, dissipation leads to a much larger absolute value of localization energy.

These findings are directly applicable to many problems involving dissipation, e. g., localization of a heavy particle on impurities in a Fermi liquid; pinning of superconducting vortices in the presence of friction etc. On the other hand, these results are only the first steps on the way to the understanding of a more complicated problem of many impurities. Since dissipation leads to stronger localization on single impurity potentials, it would be interesting to see if it also leads to changes in the theory of Anderson localization.

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